GT-PML: GENERALIZED THEORY OF PERFECTLY MATCHED LAYERS AND ITS APPLICATION TO THE REFLECTIONLESS TRUNCATION OF FINITE-DIFFERENCE TIME-DOMAIN GRIDS

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ABSTRACT

A new formulation is presented for the systematic development of perfectly matched layers (PML) from Maxwell's equations in properly constructed anisotropic media. The proposed formulation has an important advantage over the original Berenger's PML in that it can be implemented in the time domain without any splitting of the fields. Results from 3-D simulations illustrate the effectiveness of the proposed method.

INTRODUCTION

Over the past two years, Berenger's perfectly matched layer (PML) for the reflectionless truncation of differential equation-based wave simulations [1] has become the focus of extensive research. In addition to Berenger's original split-field formulation [1], as well as its interpretation in terms of a modified Maxwellian system with coordinate stretching [2], attempts have been made to avoid the need for field-splitting either in terms of the development of some type of anisotropic medium [3], or by the development of an alternative form of Berenger's equations in terms of time- and field-dependent sources [4]. These attempts have been only partially successful. The anisotropic medium approach of [3] appears to work well for frequency-domain applications; however, its application in time-domain simulation has so far been hindered by stability problems. The time-dependent source implementation of Berenger's equations reported in [4] is limited by the fact that it still requires the use of the split-field formulation at the regions where PMLs overlap.

In this paper, a generalized theory of the perfectly matched medium concept is presented that leads to the development of PMLs that can be implemented for the truncation of Finite-Difference Time-Domain (FDTD) grids without splitting of the fields.

GT-PML THEORY

We begin with the conjecture that the perfectly matched medium is an anisotropic medium with permeability and permittivity tensors given by

$$\bar{\epsilon} = \epsilon\,(\mathrm{diag}\{a, b, c\}) = \epsilon[\Lambda],$$
$$\bar{\mu} = \mu\,(\mathrm{diag}\{a, b, c\}) = \mu[\Lambda] \qquad (1)$$

where the elements of the diagonal matrix $[\Lambda] = \mathrm{diag}\{a, b, c\}$ are, in general, complex, dimensionless, constants.

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Let us define the field quantities $\hat{\mathbf{E}}$ and $\hat{\mathbf{H}}$ as follows:

$$\{\hat{E}_x, \hat{E}_y, \hat{E}_z\}^T = [G]^{-1}\{E_x, E_y, E_z\}^T,$$

$$\{\hat{H}_x, \hat{H}_y, \hat{H}_z\}^T = [G]^{-1}\{H_x, H_y, H_z\}^T \quad (2)$$

where T denotes matrix tranposition and $[G] = \mathrm{diag}\{g_x, g_y, g_z\}$ where g_x, g_y, g_z are, in general, complex constants. Choosing g_x, g_y and g_z such that

$$g_x = \sqrt{bc}, \quad g_y = \sqrt{ca}, \quad g_z = \sqrt{ab} \quad (3)$$

it can be shown that Maxwell's equations take the form

$$\nabla_{\mathbf{a}} \times \hat{\mathbf{E}} = -j\omega\mu\hat{\mathbf{H}},$$

$$\nabla_{\mathbf{a}} \times \hat{\mathbf{H}} = j\omega\epsilon\hat{\mathbf{E}},$$

$$\nabla_{\mathbf{a}} \cdot \left(\epsilon\hat{\mathbf{E}}\right) = 0,$$

$$\nabla_{\mathbf{a}} \cdot \left(\mu\hat{\mathbf{H}}\right) = 0 \quad (4)$$

where

$$\nabla_a \stackrel{\text{def}}{=} \hat{\mathbf{x}}\frac{1}{g_x}\partial_x + \hat{\mathbf{y}}\frac{1}{g_y}\partial_y + \hat{\mathbf{z}}\frac{1}{g_z}\partial_z \quad (5)$$

It can be shown that plane wave solutions of (4) are characterized by the dispersion relation $\omega^2\mu\epsilon = (k_x/g_x)^2 + (k_y/g_y)^2 + (k_z/g_z)^2$, which is satisfied by

$$k_x = kg_x \sin\theta\cos\phi,$$

$$k_y = kg_y \sin\theta\sin\phi,$$

$$k_z = kg_z \cos\theta \quad (6)$$

where $k = \omega\sqrt{\mu\epsilon}$.

To demonstrate how a perfectly matched medium can be constructed on the basis of (4), consider a two-media interface parallel to the $x - y$ plane. The fields in medium 1 satisfy (4) with material properties $\epsilon_1[\Lambda_1]$, $\mu_1[\Lambda_1]$, and corresponding stretching parameters g_{x1}, g_{y1}, g_{z1}. The fields in medium 2 satisfy (4) with material properties $\epsilon_2[\Lambda_2]$, $\mu_2[\Lambda_2]$, and corresponding stretching parameters g_{x2}, g_{y2}, g_{z2}. Through a standard reflection coefficient analysis it can be shown that the interface can be rendered reflectionless for all frequencies and all angles of incidence of a plane wave propagating, say, from medium 1 to medium 2, if

$$\epsilon_1 = \epsilon_2, \quad \mu_1 = \mu_2,$$

$$\frac{g_{x1}}{g_{y1}} = \frac{g_{x2}}{g_{y2}} \quad (7)$$

Furthermore, in view of the expression of k_z in (6), attenuation of the transmitted wave in medium 2 can be effected by proper selection of g_{z2}. Thus, a reflectionless (perfectly matched) medium is constructed. For example, for the case where medium 1 is homogeneous and isotropic it is $g_{x1} = g_{y1} = g_{z1} = 1$, and the interface will be reflectionless if the permittivities and permeabilities are the same and the elements of $[\Lambda_2]$ are such that $g_{x2} = g_{y2} = 1$. This, in view of (3), results in $a_2 = b_2$ and $c_2a_2 = 1$ which, in turn, give $g_{z2} = a_2$. If we let $a_2 = 1+(\sigma/j\omega\epsilon)$ (where we have set $\epsilon_1 = \epsilon_2 = \epsilon$, $\mu_1 = \mu_2 = \mu$), Maxwell's first curl equation inside the anisotropic perfectly matched medium becomes

$$\frac{1}{\mu}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) = -j\omega H_x - \frac{\sigma}{\epsilon}H_x \quad (8a)$$

$$\frac{1}{\mu}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) = -j\omega H_y - \frac{\sigma}{\epsilon}H_y \quad (8b)$$

$$\left(1 + \frac{\sigma}{j\omega\epsilon}\right)\frac{1}{\mu}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = -j\omega H_z \quad (8c)$$

With regards to the time-dependent form of the above equations, we observe that (8a),(8b) have the standard form for wave propagation in a lossy medium with magnetic conductivity $\sigma^* = \sigma(\mu/\epsilon)$. Transforming (8c) to the time domain we obtain

$$\frac{1}{\mu}\left(\frac{\partial E_y(t)}{\partial x} - \frac{\partial E_x(t)}{\partial y}\right)$$

$$= -\frac{\partial H_z(t)}{\partial t} - \frac{\sigma}{\epsilon}\int_0^t \left(\frac{\partial E_y(\tau)}{\partial x} - \frac{\partial E_x(\tau)}{\partial y}\right) d\tau \tag{9}$$

The integral on the right-hand side of (9) is simply the time integration of the z component of $\nabla \times \mathbf{E}$, and is interpreted as a time-dependent source present only within the perfectly matched, anisotropic absorber.

From duality it is apparent that a system similar to (8) is obtained from Maxwell's curl equation for the magnetic field. Thus, a time-dependent source term, involving the time integral of the z component of $\nabla \times \mathbf{H}$, appears in the update equation for E_z (the component of $\mathbf{E}$ normal to the interface). Thus, for perfectly matched, anisotropic media with one direction of attenuation, two time-dependent sources appear in the time-dependent form of Maxwell's equations. These results can be extended to perfectly matched, anisotropic media with more than one directions of attenuation. In all cases, there is no need for field splitting; instead, equations with dependent sources of the form of (9) occur.

NUMERICAL VALIDATION

In order to validate numerically the derived time-dependent source implementation of the anisotropic, perfectly matched medium, a z-directed point source at the center of a $50 \times 50 \times 51$-cell domain, Ω_N, was excited by a smooth compact pulse. The domain of computation was terminated by either Berenger's PML backed by perfect electric conductors, or by the proposed GT-PML also backed by perfect electric conductors. The benchmark FD-TD solution, with zero truncation boundary reflections, was obtained by simulating radiation by the aforementioned point source in a much larger domain, Ω_L, centered at the point source, discretized by a finite-difference grid of same cell size as that for Ω_N, and with truncation boundaries placed sufficiently far away to provide for causal isolation for all points in Ω_N over the time interval used for the comparisons. The error due to numerical reflections caused by the presence of the conductor-backed PMLs was obtained by subtracting at each time step the field at any grid point inside Ω_N from the field at the corresponding point in Ω_L. Fig. 1 compares the local error for $E_z(x,0,0)$ as observed at time step 100 for the standard Berenger's PML (dashed-line) to that for the proposed GT-PML (solid line). In both cases, an eight-element PML was used with quadratic variation in the conductivities. The effectiveness of the proposed GT-PML is clearly demonstrated.

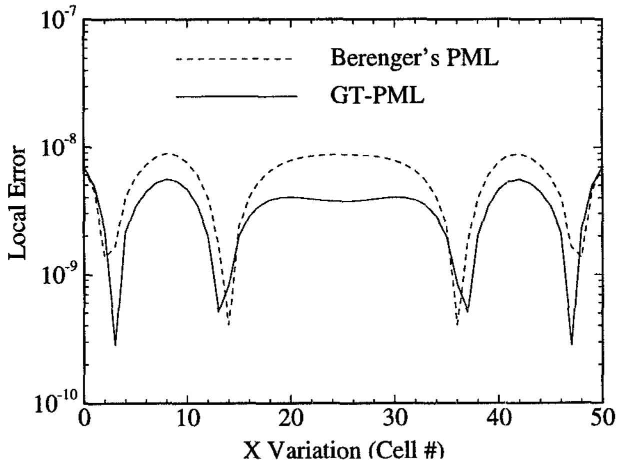

Figure 1: Local error in $E_z(x,0,0)$ within a $50 \times 50 \times 51$-cell FD-TD grid with a pulsed z-directed point source at its center. Grid truncation was effected using 8-layer Berenger's split-field PML, as well as the proposed 8-layer unsplit-field GT-PML.

CONCLUSION

A new generalized mathematical formulation has been presented for the systematic development of perfectly matched layers from Maxwell's equations in properly constructed anisotropic media. These layers can be used for numerical grid truncation in both frequency- and time-dependent wave simulations using finite-difference techniques. The proposed formulation has the advantage that it can be implemented in the time domain without any splitting of the fields.

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